The Chalk Trick Pre-selection Report

It is possible to draw continuous lines in a blackboard with chalk. However, by changing the angle of contact, the line drawn on the board becomes a dotted line, though the movement is still continuous. What parameters from the relative movement between the chalk and the board can be inferred from the resulting trace? Is it possible to infer anything about the dimensions of the chalk?

INTRODUCTION

A video of MIT professor Walter Lewin drawing dotted lines on a blackboard gained widespread popularity, yet the phenomenon remains largely omitted in scientific literature. It has been observed that it is the angle between the chalk and the blackboard that determines whether the chalk continuously stays on the surface, or starts bouncing off of it - a larger angle leads to dotted lines.

In our study, we used the slip-stick model to investigate and explain this phenomenon. The term slip-stick refers to patterns of motion where the continuous movement of an object is interrupted by self-induced vibrations caused by consecutive slipping and sticking between the surfaces ([1]). The usual components of slip-stick behaviour in mechanical systems are compliance and friction. After performing theoretical analyses based on the mechanical system shown in Figure 2 in Appendix, we adjusted the theory to refer to our experimental setup. We have conducted numerous experiments with varying relevant parameters and contrasted the results with the slip-stick theory. Based on the theory, an attempt has been made to infer information about the dimensions of the chalk and conditions of the experiment from the lines.

QUALITATIVE ANALYSIS

The so-called chalk trick can be easily explained by a *slip-stick* phenomenon. It occurs when two objects are sliding over each other. Sometimes, instead of continuous motion, they can move in a jerking motion. It is the source of a sound generated by many instruments, e.g. unique violin sound. Another very good example is a sound generated by moving a wet finger along the rim of a glass.

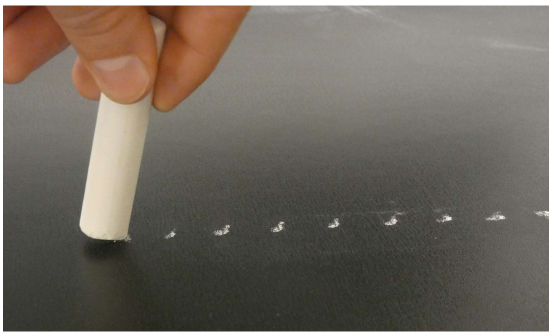

Figure 1. Dotted line - an example of the slip-stick motion with chalk and blackboard

A simple explanation of a slip-stick motion can be done using a setup with a block connected to a spring, that is placed on an inclined transmission belt (Figure 2 in Appendix). The belt moves with a constant velocity, causing the block to oscillate and constantly change friction values from kinetic to static near the amplitude's maximums.

There are three forces acting on the block — gravity force, elastic force and friction. At the beginning, when a block is placed on a belt, it starts to accelerate due to kinetic friction. When the velocity of the block is equal to the belt's velocity $v_{block} = v_{belt}$, friction becomes static. When the spring is stretched further, it forces the block to turn and go back. This process repeats, creating motion with a slip (motion relative to the belt) phase and a stick (stationary) phase.

With a chalk, the situation is different. In order to obtain dotted lines, we have to hold a chalk in a specific, tilted way. Human hand plays an important role here. It acts not like a fixed connection, but it allows the movement of the chalk. It acts with an elastic force, just like a set of springs (Figure 2). When the chalk moves relative to the blackboard, leaving a continuous line on it, kinetic friction acts on the chalk.

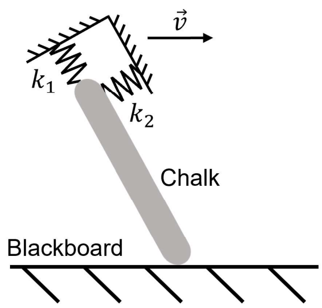

Figure 2. Simple model of a human hand

Because of the kinetic friction, chalk inclination and flexible nature of a human hand, after a short time the end of the chalk, which is touching the blackboard, will stop to move. It will result in change of the friction to static, which has bigger value, than the kinetic one. The hand, to continue the movement of the chalk, applies bigger force. When this force is bigger than the maximal static friction force, the chalk starts to move again. This leads to drastic change in friction value, because of the change to kinetic friction, and detachment of the chalk from the board. The next phase of motion is the hand moving with the chalk not touching the blackboard. Chalk is returning to its position, until it gets a contact with the blackboard. This cycle repeats, leaving discontinuous, dotted lines behind.

Although the problem refers to the video of a human drawing on the blackboard with his hand, it should not be used in the experimental setup. Humans are not capable of applying precise force nor velocity values, what leads to far bigger than acceptable measurements uncertainties. Setup with springs (Figure 2) is a good way to describe how the hand behaves, but it's not very good as an experimental setup. That's why we decided to model a human's hand with a flexible rubber beam (Figure 3). It can be easily bent, allowing the chalk to jump and has a high compression stiffness, so it's hard for the chalk to move parallel to the beam.

Figure 3. Rubber beam as an imitation of a human hand

In the case of a slip-stick motion, the transition between static and kinetic friction is essential for the phenomenon, so precise descriptions require a deep understanding of friction mechanisms. Friction origins from surface roughness. Because of roughness and applied load, contact points between surfaces deform plastically to form junctions ([2]). In order to slide one surface over the other, junctions have to be sheared, which results in friction. Static friction occurs when two solid objects are not moving relative to each other. The coefficient of static friction increases with time of contact due to creep phenomenon. We have decided to neglect this phenomenon due to short time of contact between the chalk and the blackboard. When an applied force, overcomes the force of static friction, sliding occurs. The instant sliding occurs, friction becomes kinetic. According to many studies ([3]) kinetic friction depends on the velocity of sliding. However, we neglected this, due to the small range of velocities used in experiments ($5\frac{cm}{s} - 20\frac{cm}{s}$). The sudden change of friction value, when a body starts to move, can result in slip-stick motion.

THEORETICAL MODEL

From observations of the phenomenon we know that chalk movement will consist of three different phases as indicated in Figure 4. It starts with kinetic friction decelerating the velocity of a chalk tip drawing on the blackboard until it stops and friction becomes static. Then as a hand continues to move with constant velocity, chalk rips off the surface and jumps over a short distance until it makes contact and the whole cycle repeats itself again.

In our experiments in order to obtain phenomenon that is repeatable, we decided to attach a chalk to an elastic cylinder with such Young's modulus value that

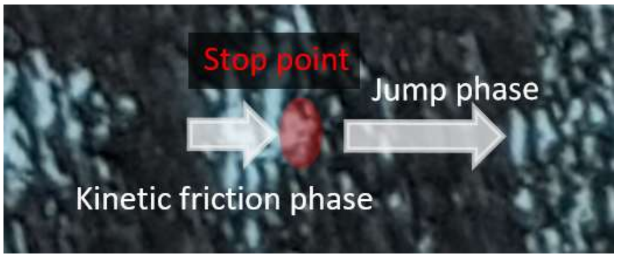

Figure 4. Different motion phases marked on top of chalk trace photo

it can bend, but it is difficult to compress it. A more elaborate description of the setup can be found in the experimental setup section. With those simplifications, we can proceed to analyse different phases of the motion of our chalk.

Static phase

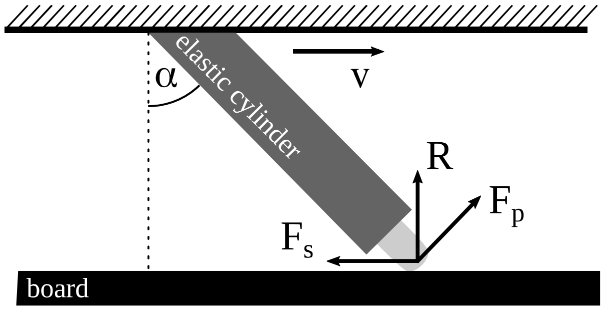

Figure 5. Scheme of our setup. Chalk is connected to an elastic cylinder, that acts like a torsion spring, with additional force pushing chalk toward the board.

We can calculate forces acting on a chalk in perpendicular direction (Figure 5). There will be some force from stiffness of a cylinder F_p and perpendicular component of a reaction force R and a static friction F_s. Value of F_s will cancel all other forces:

$$F_s \cos(\alpha) = F_p + R\sin(\alpha) \tag{1}$$

The cylinder is attached to the "arm of hand" which is moving with constant velocity along the board, causing increase in deflection of a rod. At some point, the value of F_s reaches the value of a maximal static friction. At this moment, forces acting in the opposite direction have value:

$$\mu_s R\cos(\alpha) = F_p + R\sin(\alpha) \tag{2}$$

Where: μ_s - static friction coefficient. With assumption that deflection of a cylinder is small, we know that force from it's stiffness is equal to:

$$F_p = \frac{3EI}{L^3}x = kx \tag{3}$$

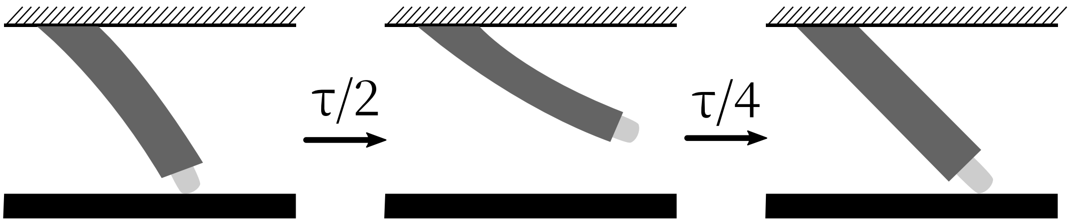

Figure 6. Our predictions of dynamics of a beam with a chalk, with marked time past each phase of a jump. After an abrupt decrease in value of friction force, chalk jumps and the beam is moving in a free vibration. It is not performing the whole period of the motion but approximately $\frac{3\tau}{4}$ because of a decompression of a cylinder

Where: x - deflection of a cylinder, E - Young's modulus, I - second moment of area, L - length of a cylinder, k - spring constant. With equations (2) and (3) and assumption that reaction force is constant because deflection of a cylinder is small, we can calculate maximal deflection at which abrupt decrease of a value of a friction force will occur:

$$x_m = \frac{R}{k}\left(\mu_s \cos(\alpha) - \sin(\alpha)\right) \tag{4}$$

Hence "arm" is moving with a constant velocity v, duration of static phase is equal to:

$$T_{static} = \frac{x_m - x_{stop}}{v} \tag{5}$$

Where: T_{static} - duration of a static phase, x_{stop} - deflection at which static phase started.

At moment of decrease of friction force value, opposing force from cylinder is much greater and it is causing chalk to loose contact with board and end static phase.

Jump phase

We will assume that when static friction switches to kinetic friction chalk looses touch with a board. If there wasn't any longitudinal compression of a cylinder, it would simply do one oscillation. However, because of a small elongation it will touch the board in the displacement equal to zero because there cylinder can be the longest 6.

If this simplification is sufficient we can now just calculate free motion of a chalk and set that the duration of a jump phase to be $\frac{3}{4}\tau$, with τ as a period of oscillation.

During jump only force acting on a cylinder come from it's internal stiffness, so the motion of a chalk is described by a free oscillations of a beam (p.115 [4]) with deflection $\xi(z,t)$:

$$EI\frac{\partial^4 \xi}{\partial z^4} = -\rho\frac{\partial^2 \xi}{\partial t^2} \tag{6}$$

Where: ρ - linear density of a cylinder. The equation is basing on an assumption that the mass of chalk is much smaller than the mass of the cylinder (mass of the cylinder - $40g$, mass of the chalk - $3g$). To calculate period of a jump phase we will assume that beam is clumped and that there is no force or torque on the end:

$$\xi|_{z=0} = 0 \quad \frac{\partial \xi}{\partial z}|_{z=0} = 0 \quad \frac{\partial^2 \xi}{\partial z^2}|_{z=L} = 0 \quad \frac{\partial^3 \xi}{\partial z^3}|_{z=L} = 0 \tag{7}$$

By solving equation (6) with boundary conditions (7) we can find frequency of a first mod of vibrations ω, which is corresponding for motion of a chalk, that is satisfying equation:

$$\cosh(\lambda\sqrt{\omega})\cos(\lambda\sqrt{\omega}) + 1 = 0 \tag{8}$$

Where: $\lambda = L\left(\frac{\rho}{EI}\right)^{1/4}$. By solving equation (8) we can find the period of a free vibration of a chalk τ.

During jump "arm" is still moving with constant value so the distance at which the chalk left no mark on the board is equal to:

$$x_{blank} = x_m + \frac{3v}{4}\tau \tag{9}$$

Drawing phase

After $\frac{3}{4}\tau$, in equations describing dynamics of beam (6) appears an additional term from kinetic friction coefficient.

$$EI\frac{\partial^4 \xi}{\partial z^4} = -\rho\frac{\partial^2 \xi}{\partial t^2} + q \tag{10}$$

Where q - load on a cylinder from kinetic friction. This additional term changes equilibrium position and set of boundary conditions:

$$\xi|_{z=0} = 0 \quad \frac{\partial \xi}{\partial z}|_{z=0} = 0 \quad \frac{\partial^2 \xi}{\partial z^2}|_{z=L} = 0 \quad \frac{\partial^3 \xi}{\partial z^3}|_{z=L} = \frac{F_k}{EI} \tag{11}$$

$$\xi|_{z=L,t=0} = 0 \quad \frac{\partial \xi}{\partial t}|_{z=L,t=0} = 2x_m\omega\cosh(\lambda\sqrt{\omega}) \tag{12}$$

Where: $F_k = \mu_k R$ - kinetic friction. By solving equation (10) with boundary conditions (11), (12) we can find time at which chalk has no velocity with respect to board. The motion of the chalk is described as:

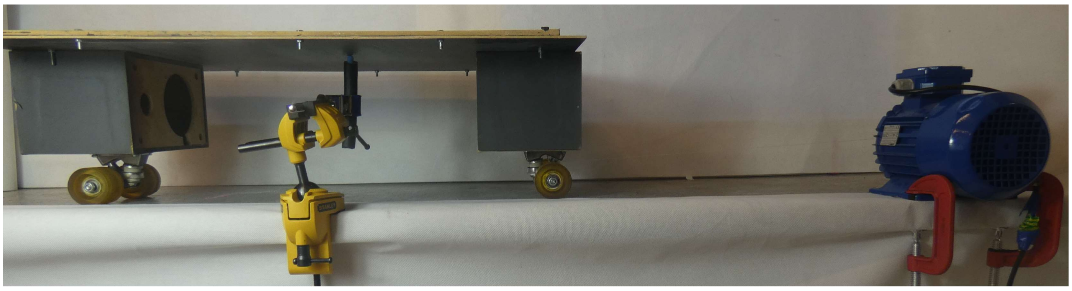

Figure 7. Experimental setup: from the left - blackboard on the wheels with black part facing downwards; movable chalk stand in the beam cylinder pressed to the chalkboard; rotating engine (blue) attached to the blackboard with non-stretchable thread

$$\xi(L,t) = \cosh(\lambda\sqrt{\omega})(2x_m \sin(\omega t) - \frac{A}{\cosh(\lambda\sqrt{\omega})}\cos(\omega t)) + A \tag{13}$$

Where: $A = \frac{F_k L^3}{3EI}$. We are looking for the time at which velocity of a chalk is equal to:

$$\frac{\partial \xi(L,\kappa)}{\partial t} = v \tag{14}$$

Where: κ - searched time, at which the distance marked by the chalk equals:

$$x_{stop} = \xi(L,\kappa) - v\kappa \tag{15}$$

After the chalk stops, static phase starts again and the cycle closes. By substituting values in our system and measured value of reaction and parameters of the beam measured in supplementary measurements we can obtain predictions for a characteristics of a trace of a chalk on a board.

EXPERIMENTAL SETUP

In order to obtain precise and repeatable results, we created an experimental setup (Figure 7) that consisted of rubber beam, movable blackboard and an engine. A rubber beam was used to hold the chalk, similarly to a human hand, as described in qualitative analysis. The rubber beam was rigidly fixed at one end, so it wasn't able to translate. In order to make our setup easier to control, we decided to move the blackboard relatively to the fixed chalk. To do so, we added wheels to the blackboard. Blackboard was moving with a given constant speed powered by our engine.

SUPPLEMENTARY EXPERIMENTS

Friction between the chalk and the blackboard was determined in an independent experiment. Experimental setup contained an inclined plane with changeable angle of inclination and chalks, which could slide on the plane. This setup allowed us to measure static and kinetic friction coefficients. In order to determine the static friction between the chalk and the blackboard, we measured the angle of inclination at which the chalk starts to move. Whereas, to determine the kinetic friction, we tracked the motion of the chalk using tracker software. We obtained the following results:

$$\mu_s = 0.62 \pm 0.03$$
$$\mu_k = 0.51 \pm 0.02$$

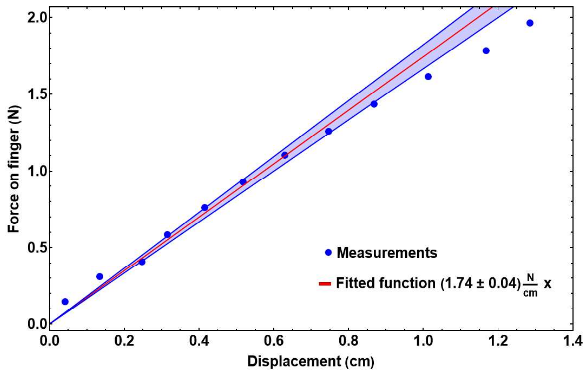

Figure 8. Measurements of vertical displacement of the end of the beam with different masses attached to it. It was approximated to be linear as displacement obtained in experiments is no larger than 0.5cm.

The quantities describing elastic beam were obtained by analysing the displacement of end of a beam after attaching mass to it. From these measurements we obtained EI, which are constant for given material and geometry. In order to increase the precision of measurements, we used two times longer beam than in the actual dotted lines experiments. The photo of experimental setup one can find in the appendix (Figure 3).

MEASUREMENTS OF CHALK TRACE

Firstly, we analysed the width of chalk trace and diameter of chalk relation during kinetic friction phase. The result obtained is that the diameter of a trace equals the part of the chalk used to draw the line. This simple approach allows to infer horizontal dimension of chalk from the line.

The next experiments were focused on the distance between dots for different angles between chalk and the board. Measurements for different angle were conducted with the same parameters of the rubber beam. It was observed that changing the parameters of beam is difficult, as the phenomenon of dotted lines occurs for the

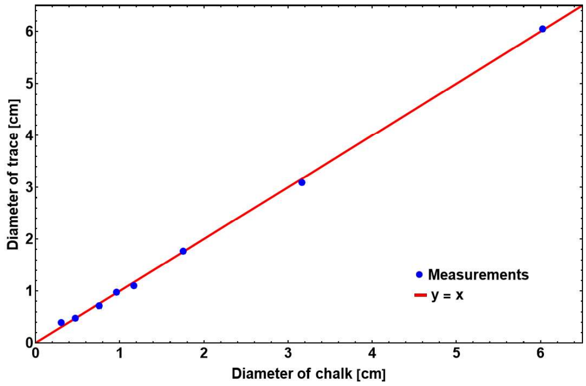

Figure 9. Measurements of the trace width for chalks of different diameters. The line represents y=x

narrow range of beam parameters. In terms of angle for our beam and velocities in the system we noticed that dotted line effect occurs only for angles between 10 and 25 degrees. When the angle is too small, the chalk remained on the blackboard. For the angles bigger than 25 degrees the chalk was sliding on the board which resulted in continuous lines. That agrees with the fact that drawing dotted lines with hand is regarded as challenging. On the (Figure 10) two cylinder-shaped chalks were used. One can see that the distance between the dots increased with increasing the angle for the same velocity of the chalk. In order to perform experimental predictions for our system we measured inclination of a beam and reaction force by placing a scale instead of a board. Each measurement point consists of 10 investigated dotted lines.

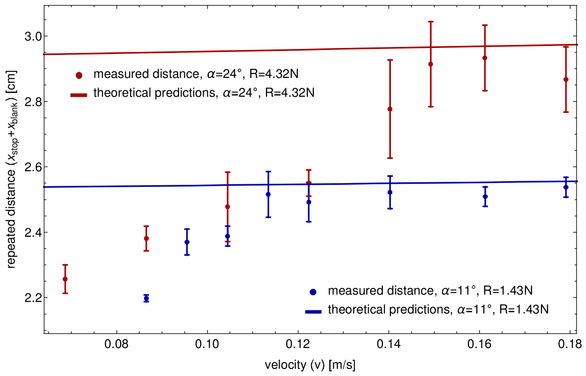

Figure 10. Measurements of distance between dots for chalks of circular cross-section. Both experiments for L= 4.2 cm

The measurements of square-shaped cross-section chalk were conducted for the one angle of inclination. Interestingly, it is more difficult to obtain the phenomenon of the chalk trick with this cross-section, both by hand and in our experimental setup. Experimental results are shown in the appendix.

RESULTS

Despite the simplicity of theoretical model derived above we were able to capture the relevant physics of the experiment at satisfactory level relatively to complexity of the human hand. When a dotted line is drawn by the human, all the relevant parameters are being changed at the same time. A hand controls both pressure on the chalk, angle, elasticity of the beam and length of the chalk, hence it is difficult model theoretically. However, with our theoretical model, knowing three of these parameters, one is able to infer information about the remaining quantity.

However, the pattern of constant length chalk jumps and traces was not always present. As can be seen in the appendix, sometimes chalk was behaving differently although all parameters seemed to be the same. In the literature [5] it was reported that similar setups can experience chaotic behaviours. It is beyond the scope of this work to investigate it further, but it could explain some of the measured abnormalities.

Conclusions and recommendations

- More advanced description of friction coefficients, as Coulomb's friction model, is too simplistic for this phenomenon. In particular time dependence of static coefficient could be important for different jumping frequencies of chalk (and thus different "static time") or speed dependency of kinetic friction coefficient what had big impact on our theoretical predictions at low speeds

- As briefly mentioned in the results, the possibility of chaotic behaviour may have been observed in rare cases in our experiments. This could be a very interesting view on investigation of the chalk trick. As literature suggests, [5] non-linear and discontinuous nature of the model used in this work with careful numerical investigation could give more insight and lead to unexpected results.

- Our measurements can be improved by investigating the sound created by the chalk hitting the blackboard. Such measurements would give data, that we could compare with our theoretical model.

[1] C. Wensrich, Slip-stick motion in harmonic oscillator chains subject to coulomb friction, Tribology International **39**, 490 (2006).

[2] M. Ashby, H. Shercliff, and D. Cebon, *Materials: Engineering, Science, Processing and Design*, Materials 3e North American Edition w/Online Testing (Elsevier Science, 2007).

[3] E. Rabinowicz, Stick and slip stick and slip, American String Teacher **11**, 18 (1961).

[4] L. D. Landau and E. M. Lifshitz, *Theory of Elasticity, Second Edition: Volume 7 (Course of Theoretical Physics)*, 2nd ed., Course of theoretical physics / by L. D. Landau and E. M. Lifshitz, Vol. 7 (Butterworth-Heinemann, 1987).

[5] M. Wiercigroch, A note on the switch function for the stick-slip phenomenon, Journal of Sound and Vibration **175**, 700–704 (1994).